Travelling wave solutions for some two-component shallow water models

Denys Dutykh, Delia Ionescu-Kruse

Denys Dutykh
CNRS–LAMA, Université Savoie Mont Blanc, France

Delia Ionescu-Kruse
Institute of Mathematics of the Romanian Academy, Bucharest, Romania

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Abstract. In the present study we perform a unified analysis of travelling wave solutions to three different two-component systems which appear in shallow water theory. Namely, we analyse the celebrated Green–Naghdi equations, the integrable two-component Camassa–Holm equations and a new two-component system of Green–Naghdi type. In particular, we are interested in solitary and cnoidal-type solutions, as two most important classes of travelling waves that we encounter in applications. We provide a complete phase-plane analysis of all possible travelling wave solutions which may arise in these models. In particular, we show the existence of new type of solutions.

Key words and phrases: travelling waves, cnoidal waves, solitary waves, Green–Naghdi model, Camassa–Holm equations, Serre equations, phase-plane analysis

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Key words and phrases. travelling waves, cnoidal waves, solitary waves, Green–Naghdi model, Camassa–Holm equations, Serre equations, phase-plane analysis.

* Corresponding author.

Contents

1. Introduction

Shallow water flows occupy a central part in hydrodynamics of free surface flows since the pioneering works of Saint-Venant [14], Boussinesq (1871 – 1877) [3, 4] and many others. The idea consists in achieving a substantial simplification of the base model (the full Euler equations, see, *e.g.* [33]) by noticing that an irrotational shallow water flow is essentially uniform across the water column. Sometimes this motion is even referred to as *columnar*. Mathematically speaking it allows to reduce the problem dimension by averaging the equations over the depth. The complexity of Boussinesq-type equations comparing to the full Euler equations is much lower. Numerous applications of shallow water models in hydraulics, coastal engineering and even in natural hazards (*e.g.* tsunamis, see [8, Ch. 7], [15]) keep the attention high on this research topic. Needless to say that every year new simplified models are proposed (however, most of them are even not properly studied deeply enough from the mathematical point of view).

In order to develop a systematic approximation procedure, one needs to characterize the full Euler equations in terms of the sizes of various parameters. The two important parameters that play a crucial role in the modern theory of water waves are ε, which measures the ratio of wave amplitude to undisturbed fluid depth, and δ, which measures the ratio of fluid depth to wavelength (see, *e.g.* [24]). The amplitude parameter ε is associated with the nonlinearity of the wave, so that small ε implies a nearly-linear wave theory. The shallowness parameter δ given by the ratio of mean water depth and wavelength, measures the deviation of the pressure, in the water below the wave, away from the hydrostatic pressure distribution.

Various approximate models, which have to describe the same physical regime, can be compared from different points of view. The main principle is that they have to be as close as possible to the corresponding solutions of the full Euler equations (closer* is better). Usually, the first comparison is done at the level of linear periodic plane wave solutions. At this level, the relationship between the wave number k and and the wave frequency ω, known as the dispersion relation, summarizes all characteristics of the model [33]. However, it is a very coarse filter, since different models can share the same dispersion relation. It comes from the prior linearisation of the system. Instead of looking at the dispersion relation, one can look, for example, at the shoaling coefficients (on a sloping beach), but this analysis is still linear [2].

Consequently, a better basis for inter-comparison should include nonlinearities. An important class of solutions which include the full nonlinear effects are the so-called *travelling waves*, *i.e.* the wave profiles propagating with a constant speed without changing their form. Localized travelling waves were discovered by Russell (1845) [28] and they are called *solitons* or *solitary waves* depending on how they interact with each other. If the shape is

*We do not intentionally specify here the notion of "closeness". The choice of the "right" functional space can be a tricky mathematical question beyond the scope of this study.

unaffected by the collision (we call this collision elastic), thus we deal with a soliton. Otherwise it is a solitary wave. Another type of travelling waves encountered in applications are periodic travelling waves, *e.g.* the cnoidal waves, derived presumably for the first time in the celebrated work of KORTEWEG & DE VRIES (1895) [25]. Consequently, in order to validate an approximate model, one can compare its travelling waves with corresponding solutions to the full EULER equations, the efficient algorithms to compute them being constantly improving [16, 35]. Since the most visible part of a water wave motion is the profile of the surface, it is natural to use this as a comparison for how effective the model is. The flow beneath the wave is very important too and we point out that the shallow water assumption (used to derive the models) leads to substantial differences here with the flow for the governing equations. For example, shallow water imposes for irrotational flows with no underlying currents a unidirectionality that is not encountered in the flow beneath the periodic travelling waves of the governing equations (see the discussions in [7, 12, 20]).

The authors have to admit that most of the studies devoted to travelling waves are focused on a particular sub-class of solutions — the so-called *solitary waves*, which decay quickly at infinity along with all their derivatives, thus justifying their name [29]. This interest has been growing since the pioneering work of ZABUSKY & KRUSKAL (1965) [37]. Moreover, for integrable equations such as KORTEWEG–DE VRIES (KdV), CAMASSA–HOLM (CH), DEGASPERIS–PROCESI (DP), *etc.*one can show rigorously using the inverse scattering technique that an arbitrary initial condition will develop into a finite number of solitary waves plus the radiation (small amplitude quasi-periodic oscillations) [9, 11, 30]. However, when one goes to the sea side, there are negligible chances to see a perfect solitary wave climbing the sloping beach. It is much more common to observe a sequence of quasi-periodic waves. That is why we consider in this study the full family of travelling wave solutions, without making any simplifying assumptions. Only when the most general equation for travelling waves is derived, we obtain the sub-family of solitary wave solutions as a particular case of a more general situation.

The present manuscript is organized as follows. The mathematical two-component shallow water models considered in this study are presented in Section 2. Section 3 is devoted to the travelling wave solutions of the models under consideration. First, we derive for each model the most general differential equation describing the whole family of such solutions. By appropriately choosing the constants of integration, we obtain the equations describing the solitary wave solutions. Some of the equations can be solved analytically to obtain the closed form explicit solutions, but we give a description of the solitary wave profiles for all models by performing a phase-plane analysis. Then, we return to the most general situation of the periodic solutions and we analyse all possible type of solutions by the phase-plane method. Finally, in Section 4 we outline the main conclusions of this study.

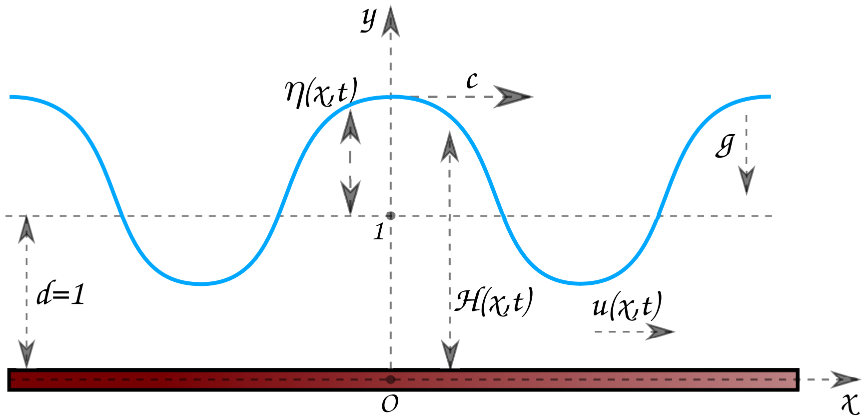

Figure 1. *The sketch of the physical fluid domain.*

2. Mathematical models

Consider a layer of an ideal incompressible homogeneous fluid with free surface. The fluid flow is assumed to be irrotational. For two-dimensional travelling waves in the fluid, the motion is identical in any direction parallel to the crest line. To describe these waves we consider a cross section of the flow that is perpendicular to the crest line, with the Cartesian coordinates $(x,\, y)$, the x-axis being in the direction of the wave propagation and the y-axis pointing vertically upwards. The flat impermeable bottom is given by $y \;=\; 0$. The fluid is acted on only by the acceleration of gravity g, the effects of surface tension are ignored. The gravity acceleration g and the mean water depth d disappear from the equations of motion by choosing appropriate dimensionless variables. The total water depth is given by the function $y \;=\; H(x,\, t) \;:=\; 1 \;+\; \eta(x,\, t)$, where $\eta(x,\, t)$ is the free surface elevation, $x \;\in\; \mathbb{R}$, $t \;\in\; \mathbb{R}^{+}$, all measured in dimensionless units. For physical reasons, the function $H(x,\, t)$ is non-negative, $H \;=\; 0$ corresponds to the solid bottom. The sketch of the fluid domain is shown in Figure 1. The variable $u(x,\, t)$ describes the horizontal velocity of the fluid, in dimensionless units.

The mathematical models considered in this study are listed below. First of all, for the sake of comparison we consider the classical Green–Naghdi (GN) equations* rediscovered independently in [19, 31, 32, 34]; for recent derivations of these equations we mention [17, 21]. The Green–Naghdi equations model shallow water waves whose amplitude is not necessarily small and can be written in the following dimensionless form:

$$
\begin{aligned}
u_t \;+\; u\,u_x \;+\; H_x \;&=\; \frac{1}{3H}\,\Bigl[\,H^3\,\bigl(u\,u_{xx} \;+\; u_{xt} \;-\; u_x^2\bigr)\,\Bigr]_x\,, \\
H_t \;+\; (\,H u\,)_x \;&=\; 0\,.
\end{aligned}
$$

*In the literature, these equations are referred to as the Serre equations, or the Su–Gardner equations but usually they are called the Green–Naghdi equations. Throughout this paper we will call them the Green–Naghdi equations.

Then, we consider an integrable two-component (CH2) generalization of the celebrated CAMASSA–HOLM equation [5], which was derived in the shallow water regime by [10, 22]:

$$\begin{aligned} u_t \;+\; 3\,u\,u_x \;-\; u_{txx} \;-\; 2\,u_x\,u_{xx} \;-u u_{xxx} +\; H H_x \;&=\; 0, \\ H_t \;+\; (\,H u\,)_x \;&=\; 0. \end{aligned}$$

And finally, we consider also another recently derived two-component system [23], which lies in-between the GN and CH2 models:

$$\begin{aligned} u_t \;+\; 3\,u\,u_x \;+\; H H_x \;&=\; \Bigl[\,H^2\,\bigl(u\,u_{xx} \;+\; u_{xt} \;-\; \tfrac{1}{2}u_x^2\bigr)\,\Bigr]_x, \\ H_t \;+\; (\,H u\,)_x \;&=\; 0. \end{aligned}$$

This model will be referred below as the new two-component (N2C) system. The details on mathematical derivations of these equations can be found in the corresponding references given above. Thus, we do not repeat them here.

3. Travelling wave solutions

Consider a family of solutions of the form:

$$H(x,t) \;=\; H(\xi), \qquad u(x,t) \;=\; u(\xi), \qquad \xi \;:=\; x \;-\; ct, \qquad c \;\in\; \mathbb{R}, \tag{3.1}$$

where c is the wave speed. Such solutions, if they exist, are called travelling waves. For a given wave amplitude the wave speed will be in general different in every model. In order to distinguish between different speeds, we shall denote them as c^{GN}, c^{CH2} and c^{N2C} correspondingly. After substituting the Ansatz (3.1) into the systems of governing equations, we obtain three systems of ODEs. The GN system becomes:

$$\begin{aligned} -c^{\mathrm{GN}}u' \;+\; uu' \;+\; H' \;&=\; \frac{1}{3H}\,\Bigl\{\,H^3\,\bigl[(u \;-\; c^{\mathrm{GN}})\,u'' \;-\; (u')^2\bigr]\,\Bigr\}', \\ \bigl(-c^{\mathrm{GN}}H \;+\; Hu\bigr)' \;&=\; 0, \end{aligned}$$

where the prime denotes the ordinary derivative with respect to the variable ξ. The CH2 system becomes:

$$\begin{aligned} -c^{\mathrm{CH2}}u' \;+\; 3uu' \;+\; c^{\mathrm{CH2}}u''' \;-\; 2u'u'' \;-\; uu''' \;+\; HH' \;&=\; 0, \\ \bigl(-c^{\mathrm{CH2}}H \;+\; Hu\bigr)' \;&=\; 0. \end{aligned}$$

And finally, the N2C system reads:

$$\begin{aligned} -c^{\mathrm{N2C}}u' \;+\; 3uu' \;+\; HH' \;&=\; \Bigl\{\,H^2\,\bigl[(u \;-\; c^{\mathrm{N2C}})\,u'' \;-\; \tfrac{1}{2}(u')^2\bigr]\,\Bigr\}', \\ \bigl(-c^{\mathrm{N2C}}H \;+\; Hu\bigr)' \;&=\; 0. \end{aligned}$$

The second equation (*i.e.* the mass conservation) can be straightforwardly integrated once:

$$u \;=\; \frac{cH \;-\; \mathcal{K}_1}{H}, \qquad \mathcal{K}_1 \;\in\; \mathbb{R}, \tag{3.2}$$

where $\mathcal{K}_1 \in \{\mathcal{K}_1^{\mathrm{GN}},\ \mathcal{K}_1^{\mathrm{CH2}},\ \mathcal{K}_1^{\mathrm{N2C}}\}$ is an integration constant and $c \in \{c^{\mathrm{GN}},\ c^{\mathrm{CH2}},\ c^{\mathrm{N2C}}\}$. The derivatives of the velocity u can be easily obtained by differentiating (3.2):

$$u' = \frac{\mathcal{K}_1 H'}{H^2}, \qquad u'' = \frac{\mathcal{K}_1 H''}{H^2} - \frac{2\mathcal{K}_1 (H')^2}{H^3}.$$

Using the expression (3.2) for u along with its derivatives, we can integrate once the momentum equations as well*:

$$3\frac{(\mathcal{K}_1^{\mathrm{GN}})^2}{H} + \frac{3}{2}H^2 = -(\mathcal{K}_1^{\mathrm{GN}})^2 H'' + \frac{(\mathcal{K}_1^{\mathrm{GN}})^2 (H')^2}{H} + \mathcal{K}_2^{\mathrm{GN}}, \tag{3.3}$$

$$-c^{\mathrm{CH2}} u + \frac{3}{2}u^2 + c^{\mathrm{CH2}} u'' = \tfrac{1}{2}(u')^2 + u u'' - \tfrac{1}{2}H^2 + \mathcal{K}_2^{\mathrm{CH2}}, \tag{3.4}$$

$$-c^{\mathrm{N2C}} u + \frac{3}{2}u^2 + \tfrac{1}{2}H^2 = -\frac{(\mathcal{K}_1^{\mathrm{N2C}})^2 H''}{H} + \frac{3}{2}\frac{(\mathcal{K}_1^{\mathrm{N2C}})^2 (H')^2}{H^2} + \mathcal{K}_2^{\mathrm{N2C}}, \tag{3.5}$$

where $\mathcal{K}_2 \in \{\mathcal{K}_2^{\mathrm{GN}},\ \mathcal{K}_2^{\mathrm{CH2}},\ \mathcal{K}_2^{\mathrm{N2C}}\} \subset \mathbb{R}$ is another integration constant. Every equation above can be multiplied by $2u' = \dfrac{2\mathcal{K}_1 H'}{H^2}$ and integrated once again lead to

$$-3\frac{(\mathcal{K}_1^{\mathrm{GN}})^3}{H^2} + 3\mathcal{K}_1^{\mathrm{GN}} H + \frac{(\mathcal{K}_1^{\mathrm{GN}})^3 (H')^2}{H^2} = 2\frac{\mathcal{K}_2^{\mathrm{GN}}(c^{\mathrm{GN}} H - \mathcal{K}_1^{\mathrm{GN}})}{H} + \mathcal{K}_3^{\mathrm{GN}}, \tag{3.6}$$

$$-c^{\mathrm{CH2}} u^2 + u^3 + c^{\mathrm{CH2}}(u')^2 - u(u')^2 = -\mathcal{K}_1^{\mathrm{CH2}} H + 2\mathcal{K}_2^{\mathrm{CH2}} u + \mathcal{K}_3^{\mathrm{CH2}}, \tag{3.7}$$

$$-c^{\mathrm{N2C}} u^2 + u^3 + \mathcal{K}_1^{\mathrm{N2C}} H = -\frac{(\mathcal{K}_1^{\mathrm{N2C}})^3 (H')^2}{H^3} + 2\mathcal{K}_2^{\mathrm{N2C}} u + \mathcal{K}_3^{\mathrm{N2C}}. \tag{3.8}$$

Finally, after some simple algebra, we obtain the following first order implicit ODEs, which describe the travelling waves, when appropriate boundary conditions are imposed. For the GN system the right-hand side is a third order polynomial in H:

$$(H')^2 = -\frac{3}{(\mathcal{K}_1^{\mathrm{GN}})^2} H^3 + \frac{\mathcal{K}_3^{\mathrm{GN}} + 2c^{\mathrm{GN}}\mathcal{K}_2^{\mathrm{GN}}}{(\mathcal{K}_1^{\mathrm{GN}})^3} H^2 - \frac{2\mathcal{K}_2^{\mathrm{GN}}}{(\mathcal{K}_1^{\mathrm{GN}})^2} H + 3. \tag{3.9}$$

In the CH2 model the right-hand side turns out to be a sixth order polynomial in H:

$$(H')^2 = H^2\Big[-\frac{1}{(\mathcal{K}_1^{\mathrm{CH2}})^2} H^4 + \frac{\mathcal{K}_3^{\mathrm{CH2}} + 2c^{\mathrm{CH2}}\mathcal{K}_2^{\mathrm{CH2}}}{(\mathcal{K}_1^{\mathrm{CH2}})^3} H^3 + \frac{(c^{\mathrm{CH2}})^2 - 2\mathcal{K}_2^{\mathrm{CH2}}}{(\mathcal{K}_1^{\mathrm{CH2}})^2} H^2 - \frac{2c^{\mathrm{CH2}}}{\mathcal{K}_1^{\mathrm{CH2}}} H + 1\Big]. \tag{3.10}$$

In the new two-component system N2C the right-hand side is a fourth order polynomial in H:

$$(H')^2 = -\frac{1}{(\mathcal{K}_1^{\mathrm{N2C}})^2} H^4 + \frac{\mathcal{K}_3^{\mathrm{N2C}} + 2c^{\mathrm{N2C}}\mathcal{K}_2^{\mathrm{N2C}}}{(\mathcal{K}_1^{\mathrm{N2C}})^3} H^3 + \frac{(c^{\mathrm{N2C}})^2 - 2\mathcal{K}_2^{\mathrm{N2C}}}{(\mathcal{K}_1^{\mathrm{N2C}})^2} H^2 - \frac{2c^{\mathrm{N2C}}}{\mathcal{K}_1^{\mathrm{N2C}}} H + 1. \tag{3.11}$$

*In some places we keep the variable u in order to have a compact notation.

The ODEs written above are most general equations which encompass all possible travelling waves of the three models under consideration in our study. So far we did not apply any additional simplifying assumptions.

3.1. Solitary waves

If we want to obtain solitary wave solutions, one has to impose additional conditions that the solitary wave profile (H, u) has to tend to a constant state $(1, 0)$ at infinity, while all the derivatives tend to $(0, 0)$, *i.e.*

$$H \;\to\; 1, \quad H^{(n)} \;\to\; 0, \quad n \;\geqslant\; 1, \quad |\,\xi\,| \to \;\infty,$$

$$u^{(n)} \;\to\; 0, \quad n \;\geqslant\; 0, \quad |\,\xi\,| \to \;\infty.$$

These boundary conditions allow us to compute exactly the integration constants $\mathcal{K}_{1,2,3}$. From the integrated mass conservation equation (3.2) we obtain (for all three models):

$$\mathcal{K}_1 \;\equiv\; c.$$

From equations (3.3)–(3.5) we find the constant $\mathcal{K}_2$:

$$\mathcal{K}_2^{\mathrm{GN}} \;=\; 3\,(c^{\mathrm{GN}})^2 \;+\; \frac{3}{2}, \qquad \mathcal{K}_2^{\mathrm{CH2}} \;=\; \frac{1}{2} \;=\; \mathcal{K}_2^{\mathrm{N2C}}.$$

Finally, from equations (3.6)–(3.8) we find $\mathcal{K}_3$:

$$\mathcal{K}_3^{\mathrm{GN}} \;=\; -3\,(c^{\mathrm{GN}})^3 \;+\; 3\,c^{\mathrm{GN}}, \qquad \mathcal{K}_3^{\mathrm{CH2}} \;=\; c^{\mathrm{CH2}}, \qquad \mathcal{K}_3^{\mathrm{N2C}} \;=\; c^{\mathrm{N2C}}.$$

By substituting the values of these constants $\mathcal{K}_{1,2,3}$ into the ODEs derived above (3.9)–(3.11), we obtain the governing equations which describe the shapes of the solitary waves. For the GN system we obtain the well-known result [32]:

$$(H')^2 \;=\; \frac{3}{(c^{\mathrm{GN}})^2}\,(H \;-\; 1)^2\,\bigl[(c^{\mathrm{GN}})^2 \;-\; H\bigr]. \tag{3.12}$$

For the CH2 system the ODE reduces to:

$$(H')^2 \;=\; \frac{1}{(c^{\mathrm{CH2}})^2}\,H^2\,(H \;-\; 1)^2\,\bigl[(c^{\mathrm{CH2}})^2 \;-\; H^2\bigr]. \tag{3.13}$$

And finally, for the N2C system the ODE for solitary waves is

$$(H')^2 \;=\; \frac{1}{(c^{\mathrm{N2C}})^2}\,(H \;-\; 1)^2\,\bigl[(c^{\mathrm{N2C}})^2 \;-\; H^2\bigr]. \tag{3.14}$$

Notice that these equations are parametrized by a single parameter — the dimensionless wave propagation speed c.

Since the left-hand side of the equations (3.12)-(3.14) is non-negative, the right-hand side must be non-negative as well. It gives us the bounds on the wave height $H(\xi)$, for $\forall \xi \in \mathbb{R}$:

$$H^{\mathrm{GN}}(\xi) \;\leqslant\; (c^{\mathrm{GN}})^2, \qquad \bigl[H^{\mathrm{CH2}}(\xi)\bigr]^2 \;\leqslant\; (c^{\mathrm{CH2}})^2, \qquad \bigl[H^{\mathrm{N2C}}(\xi)\bigr]^2 \;\leqslant\; (c^{\mathrm{N2C}})^2.$$

According to the asymptotic behaviour of H, it follows that solitary waves exist only if

$$c^2 \;>\; 1, \qquad c \;\in\; \bigl\{c^{\mathrm{GN}},\; c^{\mathrm{CH2}},\; c^{\mathrm{N2C}}\bigr\}.$$

The equation (3.12) and the equation (3.14) can be solved analytically to obtain closed form explicit solutions. They are provided in A.1. However, to our knowledge, the equation (3.13) has no such explicit solutions. From practical point of view, there are efficient algorithms (such as the Petviashvili scheme [27] and many others, see, for example, [36]) which allow to compute numerically the solution of the equation (3.14).

In what follows we give a description of the solitary wave (SW) profiles by performing a phase plane analysis for the equations (3.12)–(3.14) derived above. These equations can be put in the following general form:

$$(H')^2 \;=\; \mathcal{P}_{\mathrm{SW}}(H), \tag{3.15}$$

where $\mathcal{P}_{\mathrm{SW}}(H)$ is a polynomial function in H whose form depends on the model under consideration. From (3.15) we can easily conclude some other properties of the solitary waves. For instance, if the solitary waves exist, they are necessarily symmetric with respect to the wave crest. This property holds for the travelling wave solutions (whether solitary or periodic) of the full equations as well [13, 26]. If the wave crest is smooth, this point by definition will be the local maximum*. Consequently, the solitary wave height will be the largest real root of the polynomial $\mathcal{P}_{\mathrm{SW}}(H)$. It can be explicitly computed:

$$H^{\mathrm{GN}}_{\max} \;=\; (c^{\mathrm{GN}})^2, \qquad H^{\mathrm{CH2}}_{\max} \;=\; c^{\mathrm{CH2}}, \qquad H^{\mathrm{N2C}}_{\max} \;=\; c^{\mathrm{N2C}}.$$

The polynomials $\mathcal{P}_{\mathrm{SW}}(H)$, the corresponding phase-plane portraits and the solitary wave profiles are represented in Figures 2–4 for GN, CH2 and N2C models, respectively. The homoclinic orbits in the phase portrait lead to the pulse-type wave solutions and the heteroclinic orbits correspond to the front (or kink-type) wave solutions obtained only in the CH2 case.

3.2. Cnoidal waves

The solitary waves were analyzed above. Now we come back to a more general situation of the periodic solutions, the so-called cnoidal waves, discovered presumably in [25]. The equations (3.9)–(3.11) have the general form:

$$(H')^2 \;=\; \mathcal{P}(H), \tag{3.16}$$

where the polynomials $\mathcal{P}(H) \in \bigl\{\mathcal{P}^{\mathrm{GN}}(H),\; \mathcal{P}^{\mathrm{CH2}}(H),\; \mathcal{P}^{\mathrm{N2C}}(H)\bigr\}$,

$$\mathcal{P}^{\mathrm{GN}}(H) \;:=\; -\frac{3}{(\mathcal{K}_1^{\mathrm{GN}})^2}\,H^3 \;+\; \frac{\mathcal{K}_3^{\mathrm{GN}} \;+\; 2c^{\mathrm{GN}}\mathcal{K}_2^{\mathrm{GN}}}{(\mathcal{K}_1^{\mathrm{GN}})^3}\,H^2 \;-\; \frac{2\mathcal{K}_2^{\mathrm{GN}}}{(\mathcal{K}_1^{\mathrm{GN}})^2}\,H \;+\; 3, \tag{3.17}$$

*Indeed, for a solitary wave it will be also the global maximum.

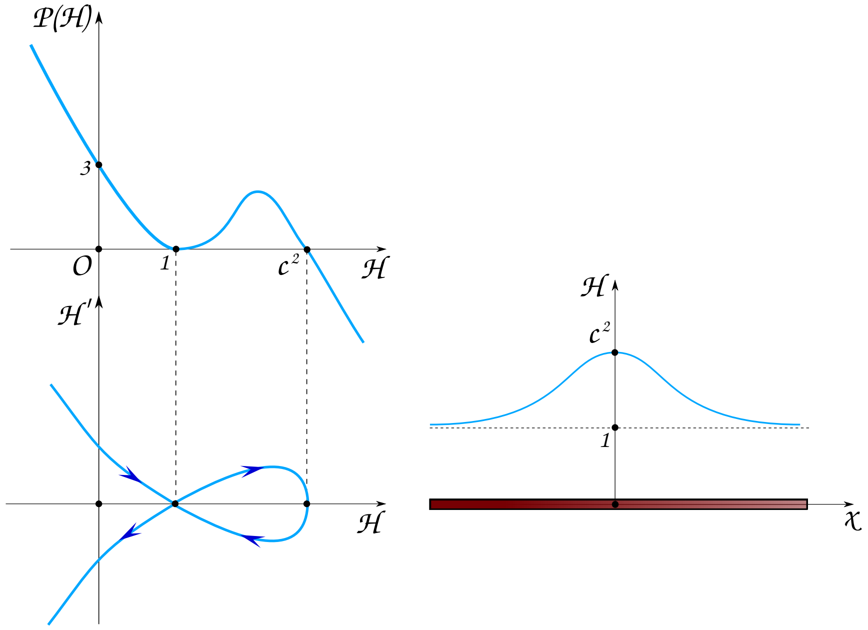

Figure 2. *The graph of the polynomial, the phase-portrait and the solitary wave profile for the GN model.*

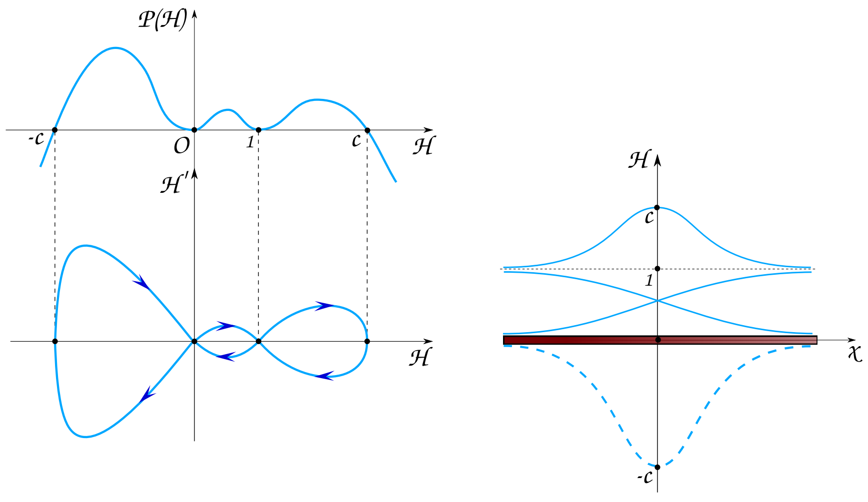

Figure 3. *The graph of the polynomial, the phase-portrait and the solitary wave profiles for the CH2 model.*

$$\begin{aligned}\mathcal{P}^{\mathrm{CH2}}(H) \;:=\; H^2\Big[&-\frac{1}{(\mathcal{K}_1^{\mathrm{CH2}})^2}\,H^4 \;+\; \frac{\mathcal{K}_3^{\mathrm{CH2}} \;+\; 2c^{\mathrm{CH2}}\mathcal{K}_2^{\mathrm{CH2}}}{(\mathcal{K}_1^{\mathrm{CH2}})^3}\,H^3 \\ &+\; \frac{(c^{\mathrm{CH2}})^2 \;-\; 2\mathcal{K}_2^{\mathrm{CH2}}}{(\mathcal{K}_1^{\mathrm{CH2}})^2}\,H^2 \;-\; \frac{2c^{\mathrm{CH2}}}{\mathcal{K}_1^{\mathrm{CH2}}}\,H \;+\; 1\Big], \qquad (3.18)\end{aligned}$$

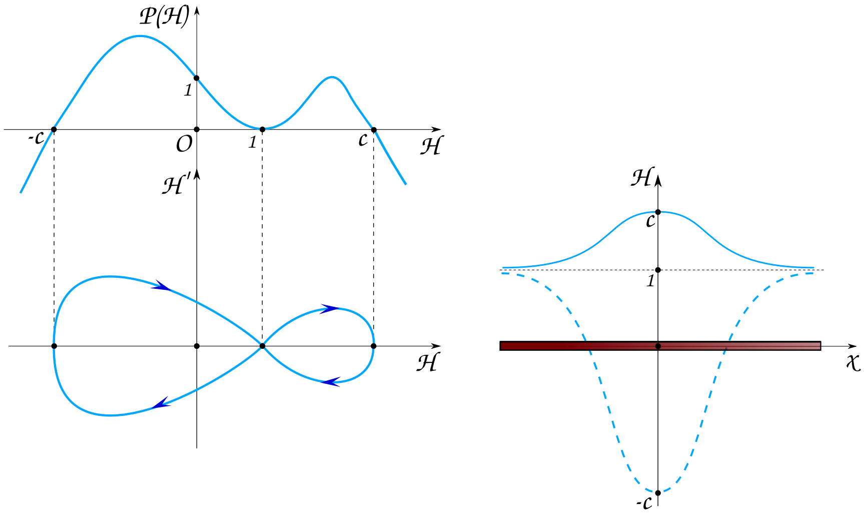

Figure 4. *The graph of the polynomial, the phase-portrait and the solitary wave profile for the N2C model.*

$$\mathcal{P}^{\mathrm{N2C}}(H) := -\frac{1}{(\mathcal{K}_1^{\mathrm{N2C}})^2}\,H^4 + \frac{\mathcal{K}_3^{\mathrm{N2C}} + 2c^{\mathrm{N2C}}\mathcal{K}_2^{\mathrm{N2C}}}{(\mathcal{K}_1^{\mathrm{N2C}})^3}\,H^3 + \frac{(c^{\mathrm{N2C}})^2 - 2\mathcal{K}_2^{\mathrm{N2C}}}{(\mathcal{K}_1^{\mathrm{N2C}})^2}\,H^2 - \frac{2c^{\mathrm{N2C}}}{\mathcal{K}_1^{\mathrm{N2C}}}\,H + 1. \quad (3.19)$$

A necessary condition for the existence of travelling waves is $\mathcal{P}(H) > 0$. With this condition fulfilled, the choice of integration constants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, yields different possible types of wave profiles in the three models under consideration.

The cubic polynomial (3.17) can have three real roots or one real root and two complex conjugate roots. Because its leading coefficient is smaller than zero and its constant term is greater than zero, by VIÈTE formulas, this polynomial has at least one positive root. For distinct roots, the only possibilities that can occur are listed below.

- If $\mathcal{P}^{\mathrm{GN}}(H)$ has three real positive roots $0 < H_1 < H_2 < H_3$, then, the graph of the the polynomial $\mathcal{P}^{\mathrm{GN}}(H)$, the corresponding phase-plane portrait and the periodic wave profile look like in Figure 5. In fact, in this case, one can find the solution of the equation (3.9) explicitly (see Appendix A.3).
- If $\mathcal{P}^{\mathrm{GN}}(H)$ has three real roots $H_1 < H_2 < 0 < H_3$, then, its graph, the corresponding phase-portrait and the solutions look like in Figure 6.
- If $\mathcal{P}^{\mathrm{GN}}(H)$ has one real positive root $H_0 > 0$ and two complex conjugate roots, then, we are in the situation represented in Figure 7. In this case, one can also find the solution of the equation (3.9) explicitly (see Appendix A.4).

The sixth order polynomial (3.18) has 0 as double root and it is written as a factorization into H^2 and a forth order polynomial with the same form as the polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$ in (3.19). The leading coefficient of $\mathcal{P}^{\mathrm{N2C}}(H)$ is smaller than zero and its constant term is greater than zero, thus, by VIÈTE formulas, this polynomial has at least one positive root

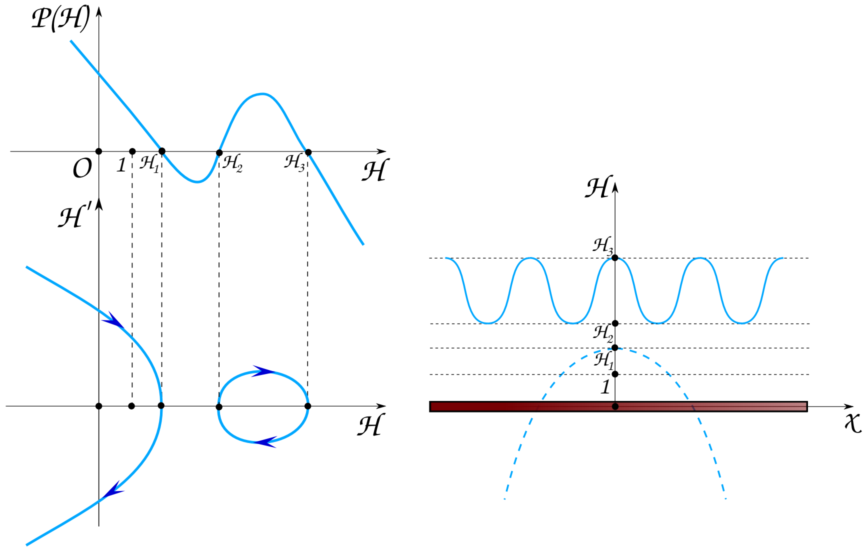

Figure 5. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{GN}}(H)$ with three real positive roots, the phase-portrait and the periodic wave profile for the GN model.*

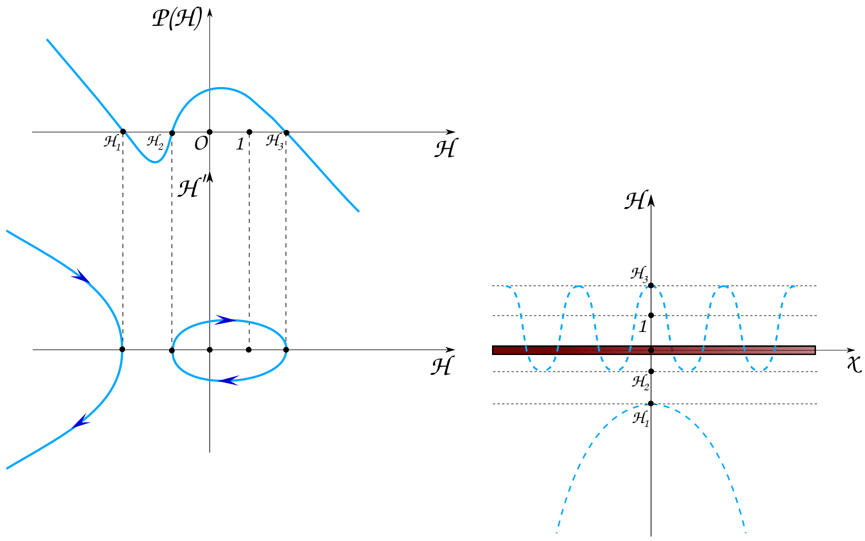

Figure 6. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{GN}}(H)$ with two negative real roots and one positive real root, the phase-portrait and the solutions for the GN model.*

and one negative root. For distinct roots, the only possibilities that can occur are listed below.

- If $\mathcal{P}^{\mathrm{N2C}}(H)$ has one negative real root H_1, one positive real root H_2 and two complex conjugate roots, then, its graph, the corresponding phase-portrait and the periodic solution look like in Figure 8. For the CH2 model, we get in this case the wave profile presented in Figure 9.

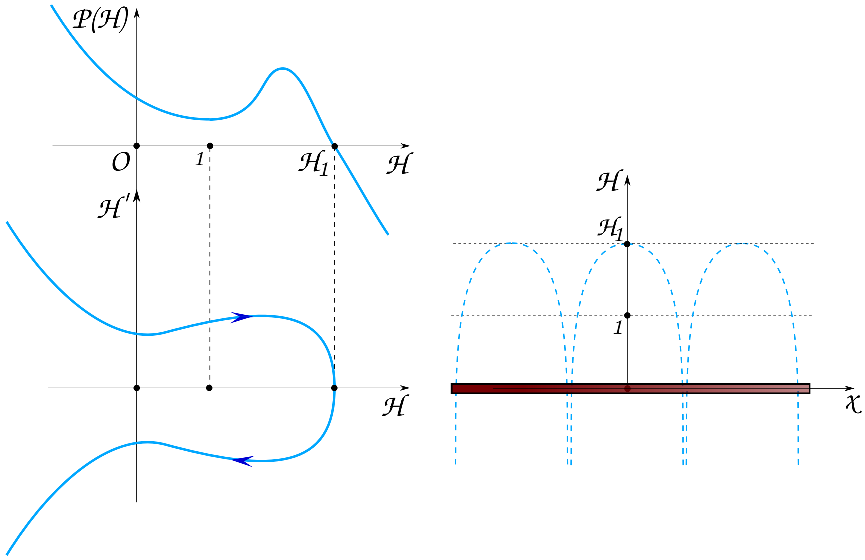

Figure 7. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{GN}}(H)$ with one real root, the phase-portrait and the periodic solution to the GN model.*

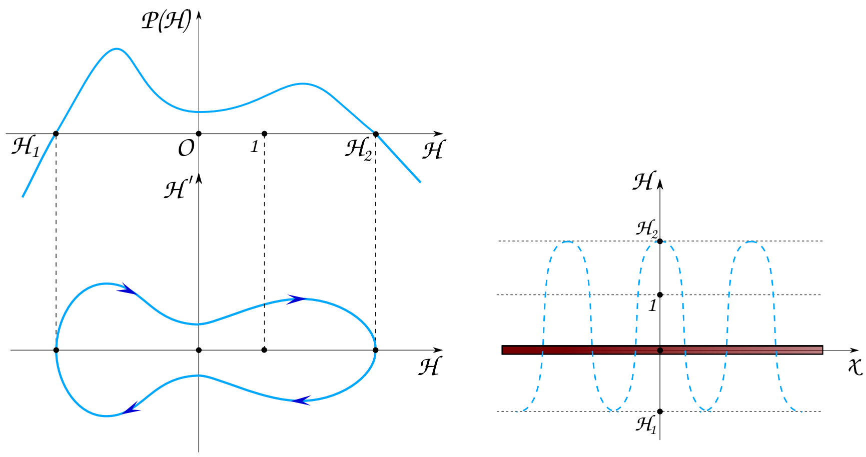

Figure 8. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$ with one negative real root, one positive real root and two complex conjugate roots, the phase-portrait and the periodic solution for the N2C model.*

- If $\mathcal{P}^{\mathrm{N2C}}(H)$ has one negative real root H_1 and three positive real roots $H_2 < H_3 < H_4$, then, its graph, the corresponding phase-portrait and the periodic wave profile look like in Figure 10. For the CH2 model, we get in this case the wave profiles presented in Figure 11.
- If $\mathcal{P}^{\mathrm{N2C}}(H)$ has three negative real roots $H_1 < H_2 < H_3$ and one positive real root H_4, then, its graph, the corresponding and the solutions look like in Figure 12. For the CH2 model, we are in the situation represented in Figure 13.

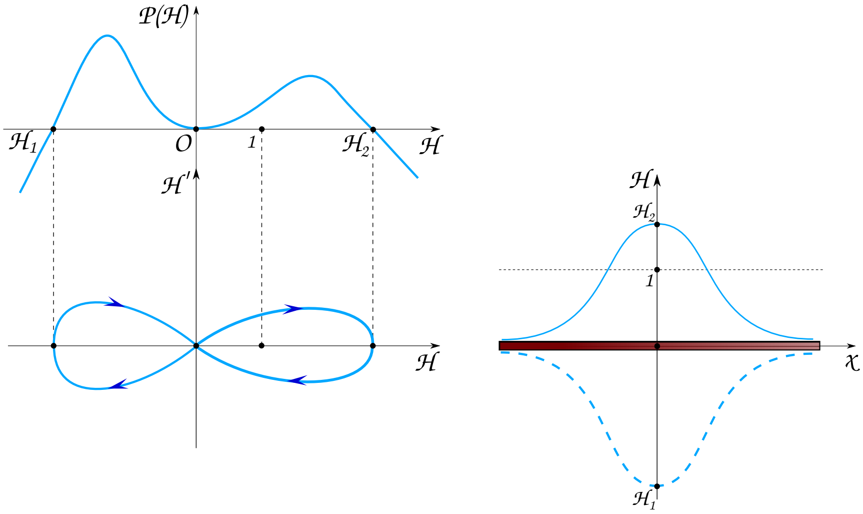

Figure 9. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{CH2}}(H)$ with one negative real root, one positive real root and two complex conjugate roots, the phase-portrait and the wave profile for the CH2 model.*

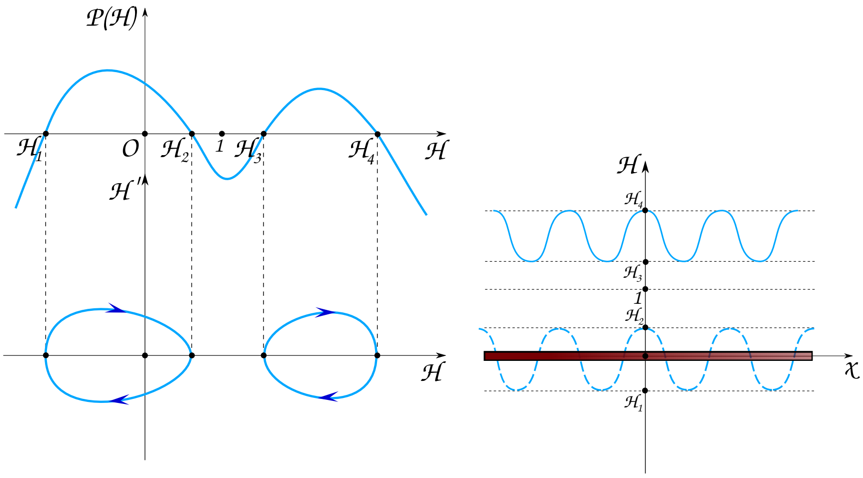

Figure 10. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$ with one negative real root and three positive real roots, the phase-portrait and the periodic wave profile for the N2C model.*

4. Conclusions

Above we considered three different two-component shallow water type models, some of them being well-known and some are new. Then, we applied a unified procedure to derive and to study analytically (whenever it was possible) the travelling wave solutions to these systems. Once the most general differential equation (ODE) describing the whole family of such solutions was derived, only then we considered the particular (and the simplest) case

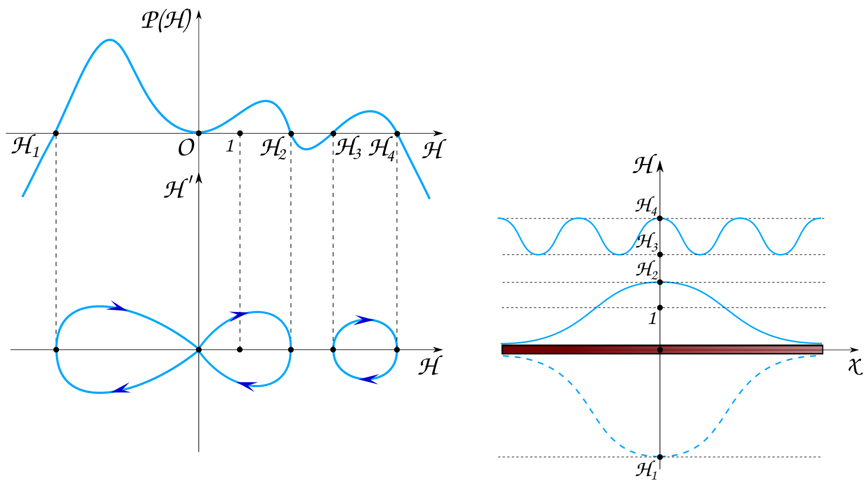

Figure 11. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{CH2}}(H)$ with one negative real root and three positive real roots, the phase-portrait and the wave profiles for the CH2 model.*

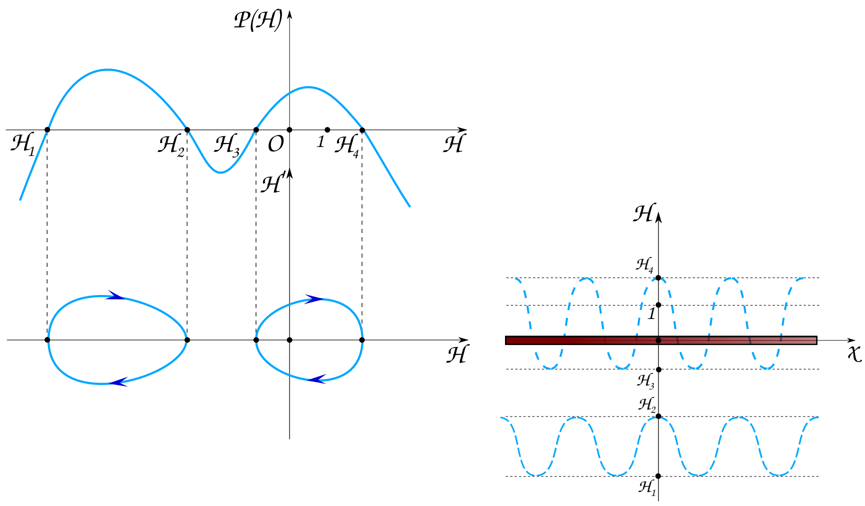

Figure 12. *The sketch of the graph of polynomial $\mathcal{P}^{\mathrm{N2C}}(H)$ with three negative real roots and one positive real root, the phase-portrait and the solutions for the N2C model.*

of solitary waves. Then, the phase-plane analysis was applied to these ODEs in order to shed some light on the behaviour of solutions, which are sometimes difficult to understand by analytical means solely. We analyzed all possible topologies of algebraic curves in the phase plane, which lead to real-valued solutions. Of course, the physically admissible solutions are those which lie above the solid bottom. In particular, we showed that the CH2 system might possess new types of solutions: front wave solutions and solutions with negative amplitude similar to anti-peakons previously found for the classical Camassa–Holm equation [1].

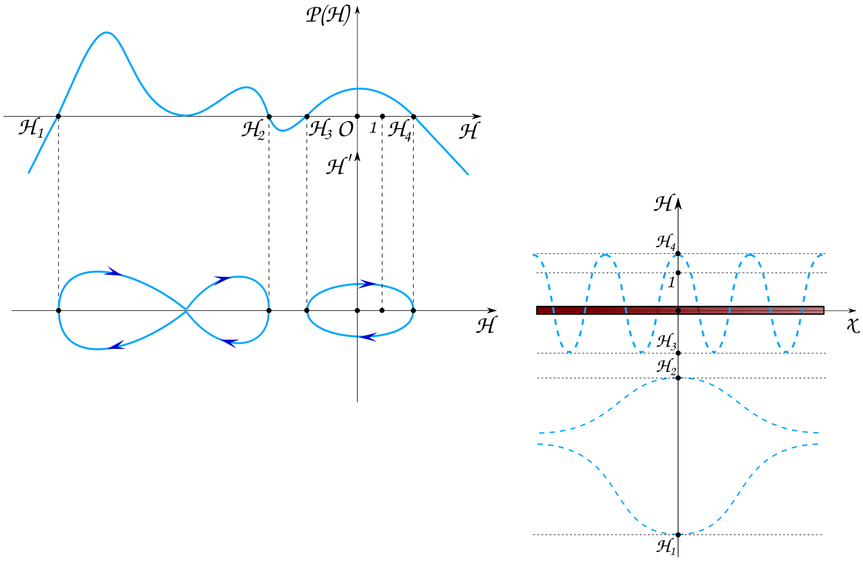

Figure 13. *The sketch of the graph of polynomial* $\mathcal{P}^{\mathrm{CH2}}(H)$ *with three negative real roots and one positive real root, the phase-portrait and the solutions for the CH2 model.*

The main particularity of our study is that we perform this analysis for three systems in parallel, thus highlighting their differences and similarities in the same place. A similar unified analytical and phase-plane approaches could be applied to other models as well, but we preferred to limit our attention to only these three systems for the sake of manuscript compactness.

A. Analytical solutions

In this Appendix we provide available analytical solutions for both types of travelling waves which arise in the three models under consideration.

A.1. Solitary waves

For the GN model, the explicit solution of the equation (3.12) is (see [32, pp. 863–864] and [34, p. 539])

$$H(\xi) \;=\; 1 \;+\; [(c^{GN})^2 - 1]\ \operatorname{sech}^2\left[\frac{\sqrt{3}}{2}\,\frac{\sqrt{(c^{GN})^2-1}}{c^{GN}}\,\xi\right]. \tag{A.1}$$

For the N2C model, the explicit solution of the equation (3.14) is (see [21])

$$H(\xi) \;=\; 1 \;+\; \frac{(c^{N2C})^2 - 1}{1 + \frac{(c^{N2C})^2+1}{2}\cosh\left[\frac{\sqrt{(c^{N2C})^2-1}}{c^{N2C}}\,\xi\right] + \frac{(c^{N2C})^2-1}{2}\sinh\left[\frac{\sqrt{(c^{N2C})^2-1}}{c^{N2C}}\,\xi\right]}\,. \qquad \text{(A.2)}$$

A.2. Cnoidal waves

The GN equations (3.12) have also the following periodic cnoidal wave solution [31] (see also [6, 18]):

$$H(\xi) \;=\; H_2 \;+\; (H_3 - H_2)\;\mathrm{cn}^2\left[\frac{\sqrt{3}}{2}\,\frac{\sqrt{H_3 - H_1}}{\mathcal{K}_1^{GN}}\;\xi;\,k\right] \qquad \text{(A.3)}$$

where $0 \;<\; H_1 \;<\; H_2 \;<\; H_3$ are the roots of the cubic polynomial (3.17), $\mathrm{cn}(\cdot\,,k)$ is the cn-Jacobi elliptic function with the elliptic modulus k, $0 \;<\; k^2 \;<\; 1$,

$$k^2 \;:=\; \frac{H_3 - H_2}{H_3 - H_1}.$$

If the cubic polynomial (3.17) has one real root, denoted $H_0 > 0$, and two complex conjugate roots, $\mathcal{P}^{GN}(H) \;=\; -\,(H - H_0)(H^2 + pH + q)$, p, $q \in \mathbb{R}$, then, the GN equations (3.12) have the following periodic solution:

$$H(\xi) \;=\; H_0 \;-\; \sqrt{H_0^2 + pH_0 + q}\;\frac{1 - \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2+pH_0+q)^{\frac{1}{4}}}{\mathcal{K}_1^{GN}}\;\xi;\,k\right]}{1 + \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2+pH_0+q)^{\frac{1}{4}}}{\mathcal{K}_1^{GN}}\;\xi;\,k\right]} \qquad \text{(A.4)}$$

with the elliptic modulus k, $0 \;<\; k^2 \;<\; 1$,

$$k^2 \;:=\; \frac{1}{2}\left(1 \;+\; \frac{H_0 + \frac{p}{2}}{\sqrt{H_0^2 + pH_0 + q}}\right).$$

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LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France

E-mail address: `Denys.Dutykh@univ-savoie.fr`

URL: `http://www.denys-dutykh.com/`

Institute of Mathematic of the Romanian Academy, Research Unit N°6, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail address: `Delia.Ionescu@imar.ro`